Volkov's Pentagon for the Modular Quantum Dilogarithm

L. D. Faddeev

St.Petersburg Department of Steklov Mathematical Institute

I. Motivation

The function $e(x)$

$$e(x) = 1 + \sum_{n=1}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{(q^{-1} - q)\dots(q^{-n} - q^{n})} x^{n} =$$

$$= \prod_{n=0}^{\infty} (1 + q^{2n+1} x)$$

has been known since Euler. The less familiar representation is given by

$$e(x) = \exp \sum_{n=1}^{\infty} \frac{x^{n}(-1)^{n}}{n(q^{n} - q^{-n})}.$$

The structure of the denominator — q-dilogarithm.

Let U and V form a Weyl pair,

$$UV = q^2 VU.$$

The following relations are true:

$$e(U)e(V) = e(U+V).$$

(Schützenberger, 1953)

$$e(V)e(U) = e(U+V+q^{-1}UV),$$

(Volkov and L.F., 1986). The pentagon relation

$$e(V)e(U) = e(U)e(q^{-1}UV)e(V).$$

Noncommutative analog of five-term relation

$$\mathsf{L}(x) + \mathsf{L}(y) = \mathsf{L}\Big(\frac{x(1-y)}{1-xy}\Big) + \mathsf{L}(xy) + \mathsf{L}\Big(\frac{y(1-x)}{1-xy}\Big)$$

for the Rogers dilogarithm (Kashaev and L.F., 1995).

Quasiclassics with use of the asymptotic

$$e(u) \sim \exp \frac{1}{2 \ln q} E(u), \qquad E(u) = \mathrm{Li}_2(-u),$$

where $\mathrm{Li}_2(u)$ is the Euler dilogarithm

$$\mathrm{Li}_2(u) = \sum \frac{u^n}{n^2}.$$

and

$$\mathrm{L}(x) = \mathrm{Li}_2(x) + \frac{1}{2} \ln x \ln(1 - x), \qquad 0 \leq x \leq 1.$$

Two questions:
1. Why does the noncommutative formula involve the Euler dilogarithm, while the Rogers dilogarithm occurs in the classical limit?
2. Why the arguments in classical and quantum formulas are so drastically different?

Partial answer in Kashaev and L.F., 1995, Kashaev, Nakamishi, 2011.
Final answer — Volkov, 2011.
For

$$R(u) = \mathsf{L}(\frac{u}{1+u})$$

the classical relation is

$$R(u) + R(v) = R(\frac{v}{1+u}) + R(\frac{uv}{1+u+v}) + R(\frac{u}{1+v}).$$

The arguments

$$x_1 = u, \quad x_2 = v, \quad x_3 = \frac{1+v}{u}, \quad x_4 = \frac{1+u+v}{uv}, \quad x_5 = \frac{1+u}{v}$$

give a solution of the Y-system $A_1 \times A_2$

$$x_i x_{i+2} = 1 + x_{i+1},$$

which has period 5

$$x_{i+5} = x_i.$$

Thus

$$R(x_1) + R(x_2) = R(\frac{1}{x_5}) + R(\frac{1}{x_4}) + R(\frac{1}{x_3}).$$

The quantum counterpart of system

$$X_i X_{i+2} = 1 + qX_{i+1}.$$

If

$$X_1 = U, \quad X_2 = V, \quad UV = q^2 VU,$$

then

$$\begin{aligned} X_3 =& U^{-1}(1 + qV), \\ X_4 =& U^{-1}(q^{-1} + U + V)V^{-1} \\ X_5 =& (1 + qU)V^{-1}. \end{aligned}$$

Volkov's quantum formula reads

$$e(X_1)e(X_2) = e(X_5^{-1})e(X_4^{-1})e(X_3^{-1}).$$

Volkov — formal series

In this talk — Hilbert space framework

$$e(x) \to \Theta(x) = \frac{\prod(1+q^{2n+1}x)}{\prod(1+\tilde{q}^{2n+1}\tilde{x})},$$

$$q = e^{i\pi\tau}, \quad \tilde{q} = e^{-i\pi/\tau}, \quad \tilde{x} = x^{1/\tau}$$

II. The modular quantum dilogarithm

Integral representation

$$\gamma(z) = \exp\{-\frac{1}{4}\int_{-\infty}^{\infty} \frac{e^{itz}}{\sin\omega t \sin\omega' t}\frac{dt}{t}\},$$

$$\omega, \omega' \text{ — periods,} \quad \tau = \frac{\omega'}{\omega}, \quad \omega\omega' = -\frac{1}{4}$$

$$\Theta(e^{-i\pi z/\omega}) = \gamma(z)$$

Function $\gamma(z)$ has a long history
Properties:

1. The functional equation

$$\frac{\gamma(z+\omega')}{\gamma(z-\omega')} = 1 + e^{-i\pi z/\omega}.$$

and similar after interchange $\omega \leftrightarrow \omega'$

In terms of $\Theta(u)$

$$\frac{\Theta(qu)}{\Theta(q^{-1}u)} = \frac{1}{1+u}.$$

2. The inversion formula

$$\gamma(z)\gamma(-z) = e^{i\beta}e^{i\pi z^2}, \quad \beta = \frac{\pi}{12}(\tau + \frac{1}{\tau}).$$

3. Unitarity

$$\overline{\gamma(z)} = \frac{1}{\gamma(z)}$$

for $\tau > 0$ and real z.

4. The quasiclassical asymptotics for $\tau \to 0$

$$\gamma(z) \sim \exp\frac{1}{2\pi i\tau}E(e^{-i\pi z/\omega}).$$

5. The pentagon equation

$$\Theta(V)\Theta(U) = \Theta(U)\Theta(q^{-1}UV)\Theta(V)$$

III. The evolution operator for the Y-system

$$Uf(z) = e^{-i\pi z/\omega} f(z), \quad Vf(z) = f(z + 2\omega'),$$
$$\tau > 0, \quad \mathrm{Re}\omega = \mathrm{Re}\omega' = 0$$

Dense domain $\mathcal{D}$: $f(z) = e^{-\alpha z^2} P(z)$

$$U^* = U, \quad V^* = V, \quad X_i^* = X_i$$

$$Kf(z) = \Theta(e^{-i\pi z/\omega}) f(z) = \gamma(z) f(z)$$

$$Ff(z) = \int_{-\infty}^{\infty} e^{-2\pi i z t} f(t) dt.$$

$$UF = FV, \quad VF = FU^{-1}$$

$$V\Theta(U) = \Theta(U)(1 + q^{-1}U)V.$$

$$S = KF$$
$$X_{i+1} = S^{-1} X_i S$$

Periodicity: $i \equiv i + 5$

$$S^5 = e^{i\alpha} I$$

Proof: Let

$$Rf(x) = e^{i\pi x^2} f(x)$$

then

$$VR = Rq^{-1} UV.$$

Indeed

$$VR = e^{i\pi(x+2\omega')^2} V = Re^{4\pi i x\omega' + 4\pi i \omega'^2} V = Re^{-i\pi x/\omega} e^{-i\pi\tau} V$$

Now we can evaluate S^5 (R. Kashaev)

$$\Theta(U)F\Theta(U)F\Theta(U)F\Theta(U)F\Theta(U)F =$$
$$\Theta(U)\Theta(V^{-1})F^2\Theta(U)F\Theta(U)F\Theta(U)F = \quad (F^3 = F^{-1})$$
$$\Theta(U)\Theta(V^{-1})\Theta(U^{-1})F^{-1}\Theta(U)F\Theta(U)F =$$
$$\Theta(U)\underbrace{\Theta(V^{-1})\Theta(U^{-1})}\Theta(V)\Theta(U)F =$$
$$\underbrace{\Theta(U)\Theta(U^{-1})}\Theta(qV^{-1}U^{-1})\underbrace{\Theta(V^{-1})\Theta(V)}\Theta(U)F =$$
$$= e^{2i\beta}R\Theta(qV^{-1}U^{-1})F^{-1}RF\Theta(U)F =$$
$$= e^{2i\beta}RF^{-1}RF\Theta(U^{-1})\Theta(U)F = e^{3i\beta}RF^{-1}RFRF$$
$$= e^{3i\beta+i\pi/4}I$$

Thus

$$\alpha = 3\beta + \frac{\pi}{4} = \frac{i\pi}{4}(\tau + \frac{1}{\tau} + 1)$$

IV. The derivation of Volkov's formula

Second form of the evolution operator

$$S = e^{i\beta}\widehat{K}^{-1}G, \quad \beta = \frac{\pi}{12}(\tau + \frac{1}{\tau}),$$

where

$$\widehat{K}f(x) = \Theta(e^{i\pi x/\omega})f(x) = \gamma(-x)f(x)$$

and

$$Gf(x) = e^{i\pi x^2}(Ff)(x) = RFf(x)$$

$$G^3 = e^{i\pi/4}F^2$$

Volkov's formula in the form

$$\Theta(X_1)\Theta(X_2) = \Theta(X_5^{-1})\Theta(X_4^{-1})\Theta(X_3^{-1})$$

or

$$KS^{-1}KS = S^{-4}\hat{K}S^4S^{-3}\hat{K}S^3S^{-2}\hat{K}S^2$$

Using

$$\hat{K}S = e^{i\beta}G, \quad S^{-1}K = F^{-1}$$

get

$$KF^{-1} = S^{-4}G^3e^{3i\beta} = e^{-i\alpha+3i\beta}SG^3 = e^{-i\pi/4}KFG^3$$

$$F^4 = I$$

V. Quasiclassical limit

The kernel of S

$$S^5(x,y) = \int M(x,z) e^{2\pi i(x-y)z} dz,$$

where

$$M(x,z) = \gamma(x)\gamma(z) \int \exp\{-2\pi(xt_1+t_1t_2+t_2t_3+t_3z+xz)\}\times \\ \times \gamma(t_1)\gamma(t_2)\gamma(t_3) dt_1 dt_2 dt_3.$$

We already know that

$$M(x,z) = e^{i\alpha}$$

Asymptotics for $\tau \to 0$

$$M(x,z) \sim \int \exp \frac{1}{2\pi i\tau}\Big\{\sum_{i=1}^{5}(E(e^{p_i}) + p_i p_{i+1})\Big\} dp_3 dp_4 dp_5,$$

where

$$p_1 = -\frac{i\pi x}{\omega}, \quad p_5 = -\frac{i\pi z}{\omega}, \quad p_{i+1} = -\frac{i\pi t_i}{\omega}, \quad i = 1,2,3.$$

The relation

$$\frac{d}{dp}E(e^p) = -\ln(1+e^p)$$

leads to phase equations

$$\begin{aligned} \ln(1+e^{p_2}) =& p_1 + p_3, \\ \ln(1+e^{p_3}) =& p_2 + p_4, \\ \ln(1+e^{p_4}) =& p_3 + p_5, \end{aligned}$$

Y-system for $x_i = e^{p_i}$, expressing x_2, x_3 and x_4 via $x_1 = x$ and $x_5 = z$.

Now

$$\sum p_i p_{i+1} = \sum \frac{1}{2} p_i (p_{i+1} + p_{i-1})$$

$$\sum \Big(E(x_i) + \frac{1}{2} \ln x_i \ln(1 + x_i) \Big) = -\frac{\pi^2}{2} = -3\frac{\pi^2}{6}.$$

Classical relation for $R(x)$ due to

$$R(x) = -E(x) - \frac{1}{2} \ln x \ln(1 + x)$$

and

$$\frac{\pi^2}{6} - R(x) = R(\frac{1}{x}).$$

Full action vs truncated action